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Euler Angles, Quaternions, and Transformation Matrices

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SHUTTLE PROGRAM

EULER ANGLES, QUATERNIONS, AND TRANSFORMATION MATRICES - WORKING RELATIONSHIPS

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By D. M. Henderson
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1.0 INTRODUCTION

Due to the extensive use of the quaternion in the onboard Space Shuttle Computer System, considerable analysis is being performed using relationships between the quaternion, the transformation matrix, and the Euler angles. This Internal Note offers a brief mathematical development of the relationships between the Euler angles and the transformation matrix, the quaternion and the transformation matrix, and the Euler angles and the quaternion. The analysis and equations presented here apply directly to current Space Shuttle problems. Appendix A presents the twelve three-axis Euler transformation matrices as functions of the Euler angles, the equations for the quaternion as a function of the Euler angles, and the Euler angles as a function of the transformation matrix elements.

The equations of Appendix A are a valuable reference in Shuttle analysis work and this Internal Note is the only known document where each of the twelve Euler angle to quaternion relationships are given. Appendix B presents a group of utility subroutines to accomplish the Euler-matrix, quaternion-matrix, and Euler-quaternion relationships of Appendix A.

2.0 DISCUSSION

2.1 Euler Angle Transformation Matrices

The following analysis and utility subroutines are offered to simplify computer programs when working with coordinate transformation matrices and their relationships with the Euler Angles and the Quaternions. The coordinate transformation matrices discussed here are defined using the following figure,

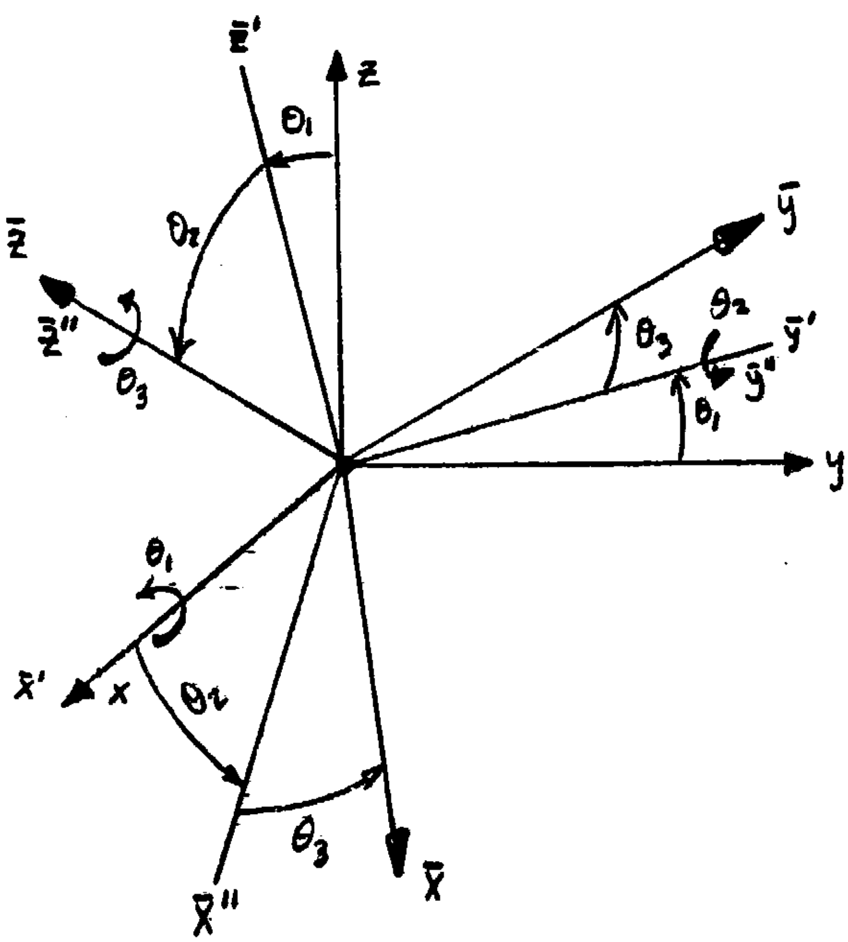

Figure 1.- Coordinate system and Euler angles.

The transformation matrix M, is defined to transform vectors in the $\bar{x}$- system ($\bar{x}$, $\bar{y}$, $\bar{z}$) into the original x-system (x, y, z) and is given by the equation,

$$x = M\bar{x}$$

where (1)

$$x = (x, y, z) \text{ and } \bar{x} = (\bar{x}, \bar{y}, \bar{z}).$$

Using the right-hand rule for positive rotations, the M matrix in (1) above is constructed by the following analysis. The first rotation in Figure 1 above is about the x-axis by the amount θ_1. The single rotation about the x-axis results in the following transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} \quad (2)$$

or $x = X\bar{x}'$ in matrix form. Rotation about the $\bar{y}'$-axis by the amount θ_2 yields the intermediate transformation matrix:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \bar{x}'' \\ \bar{y}'' \\ \bar{z}'' \end{pmatrix} \quad (3)$$

or $\bar{x}' = Y\bar{x}'$ in matrix form. Finally rotation about the $\bar{z}''$-axis by the amount θ_3 yields the intermediate transformation matrix,

$$\begin{pmatrix} \bar{x}'' \\ \bar{y}'' \\ \bar{z}'' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \qquad (4)$$

and in matrix form $\bar{x}'' = Z\bar{x}_1$. Now using the three equations,

$$\begin{aligned} x &= X\bar{x}' \\ \bar{x}' &= Y\bar{x}'' \\ \bar{x}'' &= Z\bar{x} \end{aligned} \qquad (5)$$

by substitution

$$x = (X\ Y\ Z)\ \bar{x}. \qquad (6)$$

Then from equation 1,

$$M = (X\ Y\ Z) \qquad (7)$$

Computation for the M matrix from the indicated matrix multiplication in equation (7) yields,

$$M = \begin{pmatrix} (\cos\theta_2 \cos\theta_3) & (-\cos\theta_2 \sin\theta_3) & (\sin\theta_2) \\ (\cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3) & (\cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3) & (-\sin\theta_1 \cos\theta_2) \\ (\sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3) & (\sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3) & (\cos\theta_1 \cos\theta_2) \end{pmatrix} (8)$$

The matrix M in equation (8) is a function of;

(1) The three Euler angles θ_1 θ_2 and θ_3 and

(2) The sequence of rotations used to generate the matrix.

By examination of equation (7) it is possible to show that there are twelve possible Euler rotational sequences. If the (X Y Z) notation in equation (7) represents a rotation about the X axis, then the Y axis and finally the Z axis, then the following permutations of the rotational order represents the twelve possible Euler angle sets using three rotations,

$$\begin{array}{ccc} X\ Y\ Z & Y\ X\ Z & Z\ X\ Y \\ X\ Z\ Y & Y\ Z\ X & Z\ Y\ X \\ X\ Y\ X & Y\ X\ Y & Z\ X\ Z \\ X\ Z\ X & Y\ Z\ Y & Z\ Y\ Z \end{array} \tag{9}$$

Any repeated axis rotation such as XXY does not represent a three axis rotation but reduces to the two axis rotation XY. Hence the rotations described in (9) above represent all twelve possible sets of Euler angle defining sequences. Conversely then, for a given transformation matrix there are twelve Euler angle sets which can be extracted from the matrix. The Euler matrices corresponding to all possible rotational sequences of (9) above are presented in Appendix A.

The utility subroutines "EULMAT" generates the transformation matrix from a given Euler sequence and the Euler angles. The utility subroutine "MATEUL" extracts the Euler angles from a given Euler rotational sequence. The convention is established that the Euler angles occur in the same sequences as the axis rotations. Using this concept and functional notation, equation (7) could be expressed as

$$M = X\ Y\ Z = M(\theta_x, \theta_y, \theta_z) \tag{10}$$

and from (9)

$$M = X\ Z\ X = M(\theta_x, \theta_z, \theta_x') \text{ etc.} \tag{11}$$

This convention is assumed in both subroutines and also used throughout this design note when Euler angles are used. A brief explaination of the use of these two utility subroutines is given in Appendix B.

It is interesting to note that a negative rotation in the single axis rotation matrices of equations (2), (3) and (4) will result in the formation of the transpose of the matrix. However the transpose of M in equation (7) is formed from reversing the order of multiplication and transposing the individual axis rotation equations, i.e.

$$M^T = (X\ Y\ Z)^T = (\ Y\ Z)^T X^T = Z^T\ Y^T\ X^T. \tag{12}$$

Hence the transposes of the matrices of (9) are easily formed by reversing the order of multiplication and transposing each single axis rotation equation. Using the notation in equations(10) and (11) above equation (12) could be written,

$$M^T(\theta_x, \theta_y, \theta_z) = M(-\theta_z, -\theta_y, -\theta_x). \tag{13}$$

It is recommended to avoid confusion that the forward transformation be computed and simply transposed to yield the reverse transformation matrix. All matrices of Appendix A are in the forward form, i.e. $X = M\overline{x}$ and formed from (9).

2.2 Transformation Matrices Using the Hamilton Quaternion

The transformation matrix of equation (1) can be written as a function of the Hamilton Quaternion;

$$
\begin{aligned}
q_1 &= \cos \omega/2 \\
q_2 &= \cos \alpha \sin \omega/2 \\
q_3 &= \cos \beta \sin \omega/2 \\
q_4 &= \cos \gamma \sin \omega/2 ,
\end{aligned}
\qquad (14)
$$

where ω is the rotation angle about the rotation axis with α, β, and γ direction angles with the x, y and z axes respectively. Notice also that $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The rotation angle, ω, is assumed positive according to the right-hand rule of axis rotation. The matrix M becomes

$$
M = \begin{pmatrix} (q_1^2 + q_2^2 - q_3^2 - q_4^2) & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\ 2(q_2q_3 + q_1q_4) & (q_1^2 - q_2^2 + q_3^2 - q_4^2) & 2(q_3q_4 - q_1q_2) \\ 2(q_2q_4 - q_1q_3) & 2(q_3q_4 + q_1q_2) & (q_1^2 - q_2^2 - q_3^2 + q_4^2) \end{pmatrix} . \qquad (15)
$$

For a more detailed discussion of the derivation of equation (15), see Reference 1. Using functional notation, equation (15) can be written,

$$
M = M(q_1, q_2, q_3, q_4). \qquad (16)
$$

Unlike the Euler angle rotational sequences to describe the transformation matrix of equation (1), only two quaternions can be found from equation (15). The two quaternions are:

$$\begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \quad \text{and} \quad \begin{matrix} -q_1 \\ -q_2 \\ -q_3 \\ -q_4 \end{matrix} \; . \tag{17}$$

These two quaternions represent a positive rotation about the rotation axis pointing in one direction and a positive rotation about the same line of rotation pointing in the opposite direction. Both quaternions of (17) satisfy equation (15).

The utility subroutine "QMAT" generates the transformation matrix from a given quaternion. The "QMAT" algorithm generates the matrix as given in equation (15) without duplicating any arithmetic operations. The subroutine "MATQ" extracts the positive quaternion, i.e., $q_1 > 0$, from the transformation matrix and normalizes the results to guarantee an orthogonal matrix. In order to avoid any discontinuity in extracting the quaternion from the transformation matrix, the procedure as described in Reference 2 is used.

Early works by Hamilton (Reference 3) presented the quaternion as having a scalar and a vector part, i.e.,

$$q_1 = S \qquad \vec{V} = (q_2, q_3, q_4) \tag{18}$$

and equation (16) could be expressed as,

$$M = M(q_1, q_2, q_3, q_4) = M(S, \vec{V}). \tag{19}$$

For a given quaternion the following relationship is true (from (17) above),

$$M(S, \vec{V}) = M(-S, -\vec{V}). \tag{20}$$

The transpose of the transformation matrix is given by,

$$M^T(S, \vec{V}) = M(-S, \vec{V}) = M(S, -\vec{V}). \tag{21}$$

2.3 Euler Angle and Quaternion Relationships

By examination of equations (10) and (16) the equality,

$$M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = M(\theta_1, \theta_2, \theta_3) = M(q_1, q_2, q_3, q_4) \tag{22}$$

can be written. Based on an equality for each element of the matrix the following nine equations must be true;

$$\begin{aligned}
\cos\theta_2 \cos\theta_3 &= q_1^2 + q_2^2 - q_3^2 - q_4^2 \\
-\cos\theta_2 \sin\theta_3 &= 2(q_2q_3 - q_1q_4) \\
\sin\theta_2 &= 2(q_2q_4 + q_1q_3) \\
\cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3 &= 2(q_2q_3 + q_1q_4) \\
\cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 &= q_1^2 - q_2^2 + q_3^2 - q_4^2 \\
-\sin\theta_1 \cos\theta_2 &= 2(q_3q_4 - q_1q_2) \\
\sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3 &= 2(q_3q_4 - q_1q_3) \\
\sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3 &= 2(q_3q_4 + q_1q_2) \\
\cos\theta_1 \cos\theta_2 &= q_1^2 - q_2^2 - q_3^2 + q_4^2 .
\end{aligned} \tag{23}$$

It is possible to solve for the values of the quaternion using the trigonometric half angle identities. For this Euler sequence, i.e. $X(\theta_1)\ Y(\theta_2)\ Z(\theta_3)$, the following quaternion results;

$$q_1 = -\sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2 \sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 \cos\tfrac{1}{2}\theta_3$$

$$q_2 = +\sin\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 \cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2 \sin\tfrac{1}{2}\theta_3 \cos\tfrac{1}{2}\theta_1 \tag{24}$$

$$q_3 = -\sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_3 \cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2 \cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_3$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2 \cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3 \cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2$$

Appendix A gives the quaternion as a function of the Euler angles for each of the twelve Euler rotaticnal sequence presented in Section 2.1. The special equations for the quaternion as functions of the Euler angles, like equations (24) above, are sometimes cumbersome to use, especially when multiple Euler sequences are utilized. Less coding, but perhaps more computer operations, are required by using the more general method of first generating the matrix M from the given Euler angle set and then simply extracting the quaternion from the matrix. This method is effected by first a call to "EULMAT" and then a call to "MATQ".

3.0 REFERENCES

1. Working Paper: MDTSCO, TM No. 1.4-MPB-304, E914-8A/B-003, "Quaternions and Quaternion Transformations," David M. Henderson, 23 June 1976.
2. Transmittal Memo: MDTSCO, 1.4-MPB-229, "Improving Computer Accuracy in Extracting Quaternions," David M. Henderson, 9 March 1976.
3. Sir William Rowan Hamilton, LLD, LL.D. MRIA, D.C.L. CANTAB., "Elements of Quaternions" 2 Volumes, Chelsea Publishing Company, New York, N. Y., 3rd Edition 1969, Library of Congress 68-54711 #8284-0219-1.

APPENDIX A

RELATIONSHIPS FOR THE THREE-AXIS EULER ANGLE ROTATION SEQUENCES

The twelve Euler matrices for each rotation sequence are given based on the single axis rotation equations (2), (3) and (4). The Euler matrices transform vectors from the system that has been rotated into vectors in the stationary system. Also presented here are the equations for the quaternion as a function of the Euler angles and the Euler angles as a function of the matrix elements for each rotation sequence.

(1) $M = M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = XYZ$

Axis Rotation Sequence: 1, 2, 3

$$M = \begin{bmatrix} \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 & \sin\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2 \\ -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = \sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_3 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_4 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{23}}{m_{33}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{13}}{\sqrt{1-m_{13}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{12}}{m_{11}}\right)$$

(2) $M = M(X(\theta_1), Z(\theta_2), Y(\theta_3)) = XZY$

Axis Rotation Sequence: 1, 3, 2

$$M = \begin{bmatrix} \cos\theta_2\cos\theta_3 & -\sin\theta_2 & \cos\theta_2\sin\theta_3 \\ \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_3 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{32}}{m_{22}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{12}}{\sqrt{1-m_{12}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{13}}{m_{11}}\right)$$

Axis Rotation Sequence: 1, 2, 1

$$M = \begin{bmatrix} \cos\theta_2 & \sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_3 - \sin\theta_1\cos\theta_2\sin\theta_3 & -\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3 \\ -\cos\theta_1\sin\theta_2 & +\sin\theta_1\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 & -\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = \cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = \cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_3 = \sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = \sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{21}}{-m_{31}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{12}}{m_{13}}\right)$$

(4) $M = M(X(\theta_1), Z(\theta_2), X(\theta_3)) = XZX$

Axis Rotation Sequence: 1, 3, 1

$$M = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 \\ \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = \cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = \cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_3 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = \sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{31}}{m_{21}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{13}}{-m_{12}}\right)$$

(5) $M = M(Y(\theta_1), X(\theta_2), Z(\theta_3)) = YXZ$

Axis Rotation Sequence: 2, 1, 3

$$
M = \begin{bmatrix}
\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\cos\theta_2 \\
\cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 & -\sin\theta_2 \\
\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\cos\theta_2
\end{bmatrix}
$$

$$q_1 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_3 = \sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_4 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{31}}{m_{33}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{23}}{\sqrt{1-m_{23}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{21}}{m_{22}}\right)$$

(6) $M = M(Y(\theta_1), Z(\theta_2), X(\theta_3)) = YZX$

Axis Rotation Sequence: 2, 3, 1

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_2 & \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_4 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{31}}{m_{11}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{21}}{\sqrt{1-m_{21}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{23}}{m_{22}}\right)$$

(7) $M = M(Y(\theta_1), X(\theta_2), Y(\theta_3)) = YXY$

Axis Rotation Sequence: 2, 1, 2

$$M = \begin{bmatrix} -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 \\ \sin\theta_2\sin\theta_3 & \cos\theta_2 & -\sin\theta_2\cos\theta_3 \\ -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$
$$q_2 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$
$$q_3 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$
$$q_4 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{12}}{m_{32}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{21}}{-m_{23}}\right)$$

(8) $M = M(Y(\theta_1), Z(\theta_2), Y(\theta_3)) = YZY$

Axis Rotation Sequence: 2, 3, 2

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_2\cos\theta_3 & \cos\theta_2 & \sin\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$
$$q_2 = +\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$
$$q_3 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$
$$q_4 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{32}}{-m_{12}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{23}}{m_{21}}\right)$$

(9) $M = M(Z(\theta_1), X(\theta_2), Y(\theta_3)) = ZXY$

Axis Rotation Sequence: 3, 1, 2

$$M = \begin{bmatrix} -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 \\ \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ -\cos\theta_2\sin\theta_3 & \sin\theta_2 & \cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{12}}{m_{22}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{32}}{\sqrt{1-m_{32}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{31}}{m_{33}}\right)$$

(10) $M = M(Z(\theta_1), Y(\theta_2), X(\theta_3)) = ZYX$

Axis Rotation Sequence: 3, 2, 1

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{21}}{m_{11}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{31}}{\sqrt{1-m_{31}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{32}}{m_{33}}\right)$$

(11) $M = M(Z(\theta_1), X(\theta_2), Z(\theta_3)) = ZXZ$

Axis Rotation Sequence: 3, 1, 3

$$M = \begin{bmatrix} -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 \\ \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\sin\theta_2 \\ \sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & \cos\theta_2 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$
$$q_2 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$
$$q_3 = +\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$
$$q_4 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{13}}{-m_{23}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{33}^2}}{m_{33}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{31}}{m_{32}}\right)$$

12) $M = M(Z(\theta_1), Y(\theta_2), Z(\theta_3)) = ZYZ$

Axis Rotation Sequence: 3, 2, 3

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 \\ -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 & \cos\theta_2 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_3 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{23}}{m_{13}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{33}^2}}{m_{33}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{32}}{-m_{31}}\right)$$

APPENDIX B

COMPUTER SUBROUTINES FOR THE RELATIONSHIPS

The following subroutines with a brief description of their use are presented in this appendix.

(1) "EULMAT" - Generates the transformation matrix from a given set of Euler angles and an axis rotation sequence.

(2) "MATEUL" - Extracts the Euler angles from the given transformation matrix and an axis rotation sequence.

(3) "QMAT" - Generates the transformation matrix from a given quaternion.

(4) "MATQ" - Extracts the quaternion from a given transformation matrix.

(5) "YPRQ" - Generates the quaternion directly from the yaw-pitch-roll Euler angles.

(6) "POSNOR" - Computes the positive-normalized quaternion from the given quaternion.

NAME:	EULMAT
PURPOSE:	Generates a 3 x 3 transformation matrix from a given sequence and Euler angle set.
INPUT:	ISEQ - Rotation Sequence (Integer Array (3); i.e., 1, 2, 3)
	EUL - Euler Angles in radians, in "ISEQ" Order; ARRAY (3)
OUTPUT:	A - The 3 x 3 transformation matrix
ALGORITHM REFERENCE:	Appendix A; Euler Sequences (1) thru (12).

EULER ANGLES TO THE TRANSFORMATION MATRIX

```
@FOR,IS    EULMAT,EULMAT
FOR S3E3-02/19/77-06:24:23 (,0)

    SUBROUTINE EULMAT     ENTRY POINT 000237

    STORAGE USED: CODE(1) 000250; DATA(0) 000134; BLANK COMMON(2)

    EXTERNAL REFERENCES (BLOCK, NAME)

     0003    SIN
     0004    COS
     0005    NERR3$

    STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

     0001    000105 100L     0001    000113 1??0G     0001    00001?
     0001    000142 1620     0001    000144 1650     0001    000146
     0001    000077 30L      0000 R  0000?3 P        0000 R  00005?
     0000    000057 INJP$    0000 I  000046 J        0000 I  000044
     0000 R  000047 SINA     0000 R  000053 TEMP     0000 R  000050

 00101           SUBROUTINE EULMAT(ISEQ,EUL,A)
 00103           DIMENSION ISEQ(3),EUL(3),A(3,3)
 00104           DIMENSION X(3,3,3),R(3,3)
 00105           DO 100 K=1,3
 00110           DO 10 I=1,3
 00113           DO 5 J=1,3
 00116           X(I,J,K)=0.0
 00117           IF(I.EQ.J) X(I,J,K)=1.0
 00121         5 CONTINUE
 00122        10 CONTINUE
 00125           IF(ISEQ(K).LE.0) GO TO 100
 00127           SINA=SIN(EUL(K))
 00130           COSA=COS(EUL(K))
 00131           IF(ISEQ(K).EQ.2) GO TO 20
 00133           IF(ISEQ(K).EQ.3) GO TO 30
 00135           X(2,2,K)=COSA
 00136           X(2,3,K)=-SINA
 00137           X(3,2,K)= SINA
 00140           X(3,3,K)=COSA
 00141           GO TO 100
 00142        20 X(1,1,K)=COSA
 00143           X(1,3,K)= SINA
 00144           X(3,1,K)=-SINA
 00145           X(3,3,K)=COSA
 00146           GO TO 100
 00147        30 X(1,1,K)=COSA
 00150           X(1,2,K)=-SINA
 00151           X(2,1,K)= SINA
 00152           X(2,2,K)=COSA
```

EULER ANGLES TO THE TRANSFORMATION MATRIX

(CONTINUED)

```
00153          100  CONTINUE
00155               DO 400 L=1,2
00160               M=3-L
00161               DO 300 I=1,3
00164               DO 300 J=1,3
00167               TEMP=0.0
00170               DO 250 K=1,3
00173               IF(L.EQ.1) HOLD=X(K,J,3)
00175               IF(L.EQ.2) HOLD=B(K,J)
00177               IF(ABS(HOLD).LT.1.0E-10) GO TO 250
00201               IF(ABS(X(I,K,M)).LT.1.0E-10) GO TO 250
00203               TEMP=TEMP+X(I,K,M)*HOLD
00204          250  CONTINUE
00206               IF(L.EQ.1) B(I,J)=TEMP
00210               IF(L.EQ.2) A(I,J)=TEMP
00212          300  CONTINUE
00213          400  CONTINUE
00217               RETURN
00220               END

END OF COMPILATION:         NO DIAGNOSTICS.
```

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NAME: MATEUL

PURPOSE: Extracts the Euler angles from the given transformation matrix and the required Euler rotational sequence.

INPUT: ISEQ - Rotation sequence, (Integer Array (3), i.e., 1,2,3.)

A - The 3 x 3 transformation

OUTPUT: EUL - The Euler angles, in "ISEQ" order; ARRAY(3).

ALGORITHM REFERENCE: Appendix A; Euler angles as a function of the matrix elements, sequences (1) thru (12).

TRANSFORMATION MATRIX TO THE EULER ANGLES

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```
@FOR,IS     MATEUL,MATEUL
FOR S3E3-02/19/77-06:24:25 (,0)

   SUBROUTINE MATEUL     ENTRY POINT 000335

   STORAGE USED: CODE(1) 000353; DATA(0) 000052; BLANK COMMON(2)

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003    SQRT
    0004    ATAN2
    0005    NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001     000054 10L        0001      000066 15L        0001    [illegible]
    0001     000115 30L        0001      000122 4L         0001    [illegible]
    0001     [illegible] 5L    0001      000257 6L         0001    [illegible]
    0000 R   [illegible] BSIGN 0000 R   [illegible] CSIGN  0000 R  000001
    0000 R   000003 FNUM       0000 I   000000 I           0000 I  000000
    0000 I   000001 J          0000 I   000012 JJ          0000 I  000003

00101          SUBROUTINE MATEUL(ISEQ,A,FUL)
00103          DIMENSION A(3,3),FUL(3)
00104          DIMENSION ISEQ(3)
00105          I=ISEQ(1)
00106          J=ISEQ(2)
00107          K=ISEQ(3)
00110          IECK=[illegible]
00112          IF(I.EQ.K) IECK=[illegible]
00113          BSIGN=1.0
00114          CSIGN=1.
00115          IF(I.EQ.1) GO TO 10
00117          IF(I.EQ.2) GO TO 20
00121          IF(J.NE.1) GO TO 5
00123          BSIGN=-1.0
00124          IF(IECK.NE.0) L=2
00126          GO TO 30
00127       5  CSIGN=-1.0
00130          IF(IECK.NE.0) L=1
00132          GO TO 30
00133      10  IF(J.NE.2) GO TO 15
00135          BSIGN=-1.
00136          IF(IECK.NE.9) L=3
00140          GO TO 30
00141      15  CSIGN=-1.
00143          IF(IECK.NE.0) L=2
00144          GO TO 30
00145      20  IF(J.NE.3) GO TO 25
00147          BSIGN=-1.0
```

TRANSFORMATION MATRIX TO THE EULER ANGLES
(CONTINUED)

```
00150          IF(IEOK.NE.0) L=1
00152          GO TO 30
00153       25 CSIGN=-1.0
00154          IF(IEOK.NE.0) L=3
00156       30 DO 100 N=1,3
00161          FNSGN=1.0
00162          FDSGN=1.0
00163          IF(N.EQ.2) GO TO 70
00165          IF(N.EQ.1) GO TO 50
00167          IF(IEOK.NE.0) GO TO 40
00171          FNSGN=BSIGN
00172          JJ=I
00173          GO TO 45
00174       40 JJ=L
00175          IF(BSIGN.GT.0.0) FDSGN=-1.0
00177       45 FNUM=FNSGN*A(I,J)
00200          FDEN=FDSGN*A(I,JJ)
00201          GO TO 90
00202       50 IF(IEOK.NE.0) GO TO 55
00204          FNSGN=BSIGN
00205          II=K
00206          JJ=K
00207          GO TO 60
00210       55 FDSGN=BSIGN
00211          II=L
00212          JJ=I
00213       60 FNUM=FNSGN*A(J,K)
00214          FDEN=FDSGN*A(II,JJ)
00215          GO TO 90
00216       70 IF(IEOK.NE.0) GO TO 80
00220          FNUM=CSIGN*A(I,K)
00221          FDEN=SQRT(1.0-A(I,K)**2)
00222          GO TO 90
00223       80 FNUM=SQRT(1.0-A(I,I)**2)
00224          FDEN=A(I,I)
00225       90 EUL(N)=ATAN2(FNUM,FDEN)
00226      100 CONTINUE
00230          RETURN
00231          END
```

END OF COMPILATION: NO DIAGNOSTICS.

NAME: QMAT

PURPOSE: Generates the transformation matrix from the given quaternion.

INPUT: Q - The quaternion; ARRAY(4).

OUTPUT: A - The 3 x 3 transformation matrix

ALGORITHM REFERENCE: Equation (15) from Section 2.2.

QUATERNION TO THE TRANSFORMATION MATRIX

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```
@FOR,IS      QMAT,QMAT
FOR S9E3-02/19/77-06:24:19 (,0)

   SUBROUTINE QMAT        ENTRY POINT 000077

   STORAGE USED: CODE(1) 000103; DATA(0) 000015; BLANK COMMON(2) 

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003    NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0000     000007 INJP$        0000 R 000002 P2              0000 R 000001
    0000 R  000004 P5           0000 R 000005 TEMP


                              SUBROUTINE QMAT(Q,A)
                              DIMENSION Q(4),A(3,3)
                              P2=Q(2)+Q(2)
                              P3=Q(3)+Q(3)
                              P4=Q(4)+Q(4)
                              P5=P2*Q(2)
                              P6=P4*Q(4)
                              TEMP=1.0-P3*Q(3)
                              A(1,1)=TEMP-P6
                              A(2,2)=1.0-P5-P6
                              A(3,3)=TEMP-P5
                              P5=P2*Q(3)
                              P6=P4*Q(1)
                              A(1,2)=P5-P6
                              A(2,1)=P5+P6
                              P5=P2*Q(4)
                              P6=P3*Q(1)
                              A(1,3)=P5+P6
                              A(3,1)=P5-P6
                              P5=P3*Q(4)
                              P6=P2*Q(1)
                              A(2,3)=P5-P6
                              A(3,2)=P5+P6
                              RETURN
                              END

         END OF COMPILATION:          NO DIAGNOSTICS.
```

NAME: MATQ

PURPOSE: Extracts the positive quaternion from the given transformation matrix.

INPUT: A - The 3 x 3 transformation matrix

OUTPUT: Q - The positive quaternion; ARRAY(4).

ALGORITHM REFERENCE: See Reference 2.

TRANSFORMATION MATRIX TO THE QUATERNION

```
@FOR,SI  MATQ,MATQ
FOR S0E3-02/19/77-06:24:21 (,0)

SUBROUTINE MATQ      ENTRY POINT 000203

STORAGE USED: CODE(1) 000220; DATA(0) 000056; BLANK COMMON(2)

EXTERNAL REFERENCES (BLOCK, NAME)

 0003   SQRT
 0004   NERR2$

STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

00101      SUBROUTINE MATQ(A,Q)
00103      DIMENSION A(3,3),Q(4),T(4)
00104      I=0
00105      BIG=0.0
00106      DO 40 J=1,4
00111      Q(J)=0.0
00112      IF(J.EQ.2) GO TO 10
00114      IF(J.EQ.3) GO TO 20
00116      IF(J.EQ.4) GO TO 30
00120      Q(J)=1.0
00121      TEMP=A(1,1)+A(2,2)+A(3,3)+1.0
00122      T(J)=1.0
00123      GO TO 35
00124   10 TEMP=A(1,1)-A(2,2)-A(3,3)+1.0
00125      T(J)=A(3,2)-A(2,3)
00126      GO TO 35
00127   20 TEMP=-A(1,1)+A(2,2)-A(3,3)+1.0
00130      T(J)=A(1,3)-A(3,1)
00131      GO TO 35
00132   30 TEMP=-A(1,1)-A(2,2)+A(3,3)+1.0
00133      T(J)=A(2,1)-A(1,2)
00134   35 IF(TEMP.LT.BIG) GO TO 40
00136      BIG=TEMP
00137      I=J
00140   40 CONTINUE
00142      IF(I.EQ.0) GO TO 60
00144      Q(I)=0.5*SQRT(BIG)
00145      IF(I.NE.1) Q(1)=ABS(0.25*T(I)/Q(I))
00147      TEMP=0.25/Q(1)
00151      DO 50 J=2,4
00153      Q(J)=TEMP*T(J)
00154   50 CONTINUE
00156   60 RETURN
00157      END

END OF COMPILATION:        NO DIAGNOSTICS.
```

NAME: YPRQ

PURPOSE: Generates the quaternion directly from the yaw-pitch-roll Euler angles, i.e., a 3, 2, 1 Euler sequence.

INPUT: YPR - The yaw-pitch-roll Euler angles; ARRAY (3).

OUTPUT: QO - The positive quaternion, ARRAY (4).

ALGORITHM REFERENCE: Appendix A, the quaternion equations for Euler sequence (10), a 3, 2, 1 sequence.

NOTE: This subroutine calls "POSNOR" to take the normal of the positive quaternion.

YAW-PITCH-ROLL EULER ANGLES TO THE QUATERNION


```
@FOR,S.     YPRQ,YPRQ
FOR S0E3-02/19/77-06:24:03 (,0)

   SUBROUTINE YPRQ        ENTRY POINT 000114

   STORAGE USED: CODE(1) 000121; DATA(0) 000024; BLANK COMMON(2) 00

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003     POSNOR
    0004     COS
    0005     SIN
    0006     NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0000 R 000001 CP          0000 R 000011 CR          0000 R 000007 [illegible]
    0000 R 000004 HY          0000   000016 INJP$       0000 R 000000 [illegible]
    0000 R 000012 SY


 00101            SUBROUTINE YPRQ(YPR,Q0)
 00103            DIMENSION YPR(3),Q(4),Q0(4)
 00104            HY=0.5D0*YPR(1)
 00105            HP=0.5D0*YPR(2)
 00106            HR=0.5D0*YPR(3)
 00107            CY=COS(HY)
 00110            CP=COS(HP)
 00111            CR=COS(HR)
 00112            SY=SIN(HY)
 00113            SP=SIN(HP)
 00114            SR=SIN(HR)
 00115            Q(1)=CY*CP*CR+SY*SP*SR
 00116            Q(2)=CY*CP*SR-SY*SP*CR
 00117            Q(3)=CY*SP*CR+SY*CP*SR
 00120            Q(4)=-CY*SP*SR+SY*CP*CR
 00121            CALL POSNOR(Q,Q0)
 00122            RETURN
 00123            END

     END OF COMPILATION:        NO  DIAGNOSTICS.
```

NAME: POSNOR

PURPOSE: To output the positive and normalized quaternion from the given quaternion.

INPUT: Q - The quaternion; ARRAY (4).

OUTPUT: QO - The positive-normalized quaternion; ARRAY (4).

ALGORITHM REFERENCE:

1. If the sign of Q(1) is negative:

   Set QO(I) = -Q(I) for I = 1, 2, 3, 4.

2. Set QO(I) = QO(I)/TEMP

   where TEMP = $\sqrt{QO_1^2 + QO_2^2 + QO_3^2 + QO_4^2}$

SELECTS THE POSITIVE QUATERNION AND NORMALIZES

```
@FOR,IS     POSNOR,POSNOR
FOR S3E3-02/19/77-06:24:14 (,0)

   SUBROUTINE POSNOR     ENTRY POINT 000055

   STORAGE USED: CODE(1) 000067; DATA(0) 000017; BLANK COMMON(2) 0

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003    SQRT
    0004    NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001    000016 111G        0001    000036 121G        0000 I 000002
    0000 R 000000 TEMP


00101       1*              SUBROUTINE POSNOR(Q,QQ)
00103       2*              DIMENSION Q(4),QQ(4)
00104       3*              TEMP=1.0
00105       4*              IF(Q(1).LT.0.0) TEMP=-1.0
00107       5*              SUM=0.0
00110       6*              DO 50 I=1,4
00113       7*              QQ(I)=TEMP*Q(I)
00114       8*              SUM=SUM+QQ(I)*QQ(I)
00115       9*           50 CONTINUE
00117      10*              TEMP=1.0/SQRT(SUM)
00120      11*              DO 100 I=1,4
00123      12*              QQ(I)=TEMP*QQ(I)
00124      13*          100 CONTINUE
00126      14*              RETURN
00127      15*              END

      END OF COMPILATION:        NO  DIAGNOSTICS.
```

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